STOKES PARTICLE IN AN INHOMOGENEOUS TURBULENT STREAM

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The motion of a small (Stokes) particle suspended in an inhomogeneous turbulent, incompressible viscous stream is considered. It is shown that the nonlinear interrelationship between the disordered vibrational and translational particle motions in an inhomogeneous field of turbulent pulsations specifies considerable systematic particle displacement and the transformation of the turbulent medium motion into directed motion. An estimate is made of the effects for a particle suspended in a medium whose pulsation field is stationary and varies in one direction.

As is known from model problems of mechanics [1] and plasma physics [2], which will receive greater development later, the nonlinearity of the forced vibrational motion of a particle in a time-periodic force field whose intensity depends on the coordinates specifies the appearance of average force fluctuations with respect to the period, which tend to displace a free particle in the domain where the force field intensity is lowered. It should be expected that an analogous effect is exerted on a particle submerged in a turbulent fluid or gas by a field of inhomogeneous turbulent pulsations. Let us note that the turbulent stream is always inhomogeneous, such is the manner of its existence. The pulsation field in boundary layers separating the turbulent zone from the rest flow zone or from solid boundaries is sharply inhomogeneous, and here the most noticeable systematic particle displacement should be expected. This is seen in the rapid covering of the blades of a rotating fan with dust and in other cases. Since a large circle of problems (technological, atmospheric pollution, etc.) is associated with questions of particle precipitation in fluid and gas streams, it is expedient to estimate the systematic effect mentioned.*

1. EQUATIONS OF MOTION

Let us consider a small spherical particle of diameter d and density ρ_0 suspended in an incompressible turbulent medium of density ρ and kinematic viscosity ν. As $d \to 0$ the particle trajectory is described by the equation (see Secs. 5 and 7 in [7], for instance)

$$\ddot{\mathbf{r}} + a\dot{\mathbf{r}} = a\mathbf{u} + b\mathbf{u}' + c\int_{-\infty}^{t} \frac{d\tau}{\sqrt{t-\tau}}(\mathbf{u}' - \ddot{\mathbf{r}})_\tau + \mathbf{f}_l, \tag{1}$$

where

$$a = (12\nu/d^2)b;\ b = [3\rho/(2\rho_0 + \rho);\ c = (6b/d)\sqrt{\nu/\pi};$$

$\mathbf{u}(\mathbf{r}, t)$ and $\mathbf{u}'(\mathbf{r}, t) = d\mathbf{u}/dt + (\mathbf{u}\nabla)\mathbf{u}$ are the unperturbed velocity and acceleration of the medium at the time t at the location of the particle. The coefficient a characterizes the Stokes friction force, and the term $b\mathbf{u}'$ is due to the pressure gradient in a stream unperturbed by particles. The third member is the "Basset force" which takes account of the nonuniformity in the relative particle motion, and $\mathbf{f}_l$ is the density of the external forces acting on the particle, including gravity. Equation (1) is used for estimates of $d < l$, where l is the inner scale of turbulence (at distances l the velocity drop in the stream u_l is such that $lu_l/\nu \sim 1$), and it will be more exact, the smaller the ratio d/l.

*A set of investigations has been performed on questions of particle transport and deposition in inhomogeneous turbulent streams (see [3-6] and the references cited there, for example). However, the factor of dynamic interrelation between the disordered vibrational particle motion and its average translational motion is inadequately discussed in the literature, in our opinion.

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Since the velocity field **u(r, t)** is characterized by a whole spectrum of time and space scales, (1) is actually a nonlinear stochastic equation. It conserves its nonlinear character even in the case of homogeneous isotropic turbulence. Moreover, it is impossible, for example, to consider that the effect of the force $\mathbf{f}_l$ yields simply an additive contribution to the resultant motion. Usually (see [8], for example, and Secs. 5 and 7 in [7] and the references cited there), **u** and **u'** are assumed random functions of time in analyzing (1), while their dependence on the coordinates is left outside the field of view. Hence, (1) becomes linear and admits of the direct determination of all correlation characteristics of the particle motion in terms of the turbulent stream characteristics. In particular, the particle diffusion and the reverse influence of the particles on the stream are traditionally considered on this basis.

It is conceivable that taking account of the nonlinearity in (1) in problems where effects associated with the spatial inhomogeneity of the field of turbulent pulsations are investigated specially is important in principle.

Interaction between the medium and the translational degrees of freedom of the particle is described by (1). Rotational particle motion relative to the medium also occurs in an inhomogeneous stream. The flow perturbations which hence originate specify deflections in the trajectory **r(t)** in addition to those taken into account in (1). The effect is related to the action of the Coriolis force. As $d \to 0$, the additional deflections are slight, but taking account of the rotation factor in the consideration of the average motion results in effects of the same order in d/l as the model (1) yields, as will be shown. On a sphere which moves translationally with velocity **v** and rotates simultaneously with the velocity **Ω** in a fluid at rest, a force transverse in **Ω** and **v** acts which equals [9]

$$\mathbf{F}_\Omega = k\rho d^3[\Omega \times \mathbf{v}] \tag{2}$$

for flows creeping around the sphere, where $k \approx \pi/8$. The experiments (see [10], for example) show the applicability of (2) even for particle rotation in inhomogeneous laminar modes (for Couette flow). It is natural to assume the validity of (2) for particles in a turbulent stream for which the condition $d < l$ is satisfied by understanding **v** and **Ω** to be the relative translational and rotational particle motions, i.e.,

$$\mathbf{v}(\mathbf{r},\ t) = \dot{\mathbf{r}} - \mathbf{u},\ \Omega(\mathbf{r},t) = \dot{\varphi} - \psi,\ \psi = (1/2)\,\mathrm{rot}\,\mathbf{u},$$

where φ is the angular velocity of the particle. It is convenient to convert the force $\mathbf{F}_\Omega$ to unit mass of the sphere. Taking account of the apparent mass, we obtain

$$\mathbf{f}_\Omega \approx (1/2)b[\Omega \times \mathbf{v}]. \tag{3}$$

The forces $\mathbf{f}_\Omega$ must be included in the right side (1). To determine them, let us take into account that the moment of the forces acting on a nonuniformly rotating sphere in a fluid at rest as $d \to 0$ equals [11]

$$\mathbf{M} = -(\pi/12)\rho d^5\dot{\Omega} - \pi\rho d^3\nu\Omega.$$

The sphere moment of inertia equals $(\pi/60)\,\rho_0 d^5$. Hence, an equation similar to (1) in structure

$$\dot{\varphi} + \alpha\varphi = \alpha\psi + \beta\psi' + \mathbf{m}_b + \mathbf{m}_l, \tag{4}$$

can be written for the rotational motion, where

$$\alpha = (10\nu/d^2)\beta;\ \beta = 6\rho/(\rho_0 + 5\rho);$$

$\psi(\mathbf{r}, t)$ and $\psi'(\mathbf{r}, t) = d\psi/dt + (\mathbf{u}\nabla)\psi$ are the unperturbed angular velocity and acceleration of the medium at the particle location at the time t, $\mathbf{m}_b$ is the rotational analog of the Basset force, and $\mathbf{m}_l$ is the moment of the external forces and the forces associated with the reactions to the force f_Ω. Since the effects of interaction between the translational and rotational motions at the times $\sim 1/\Omega$ are small as $d \to 0$ compared to the effect of the Stokes ($\sim a, \alpha$) and inertial ($\sim b, \beta$) forces, then the effect of the forces $\mathbf{f}_\Omega$ and their reactions cannot be taken into account in (1) and (4) in analyzing the motions at the times $\sim 1/\Omega$. Assuming the motions independent, let us take the average of the right side of (1) and the forces (3) and their total effect and let us determine the systematic particle displacement. Let us neglect the Basset force and the momentum $\mathbf{m}_b$ in the analysis. Taking them into account makes the exposition awkward, but introduces no principal changes in the result for small d. Let us also assume $\mathbf{f}_l = 0$ and $\mathbf{m}_l = 0$. Let us also assume $\mathbf{f}_l \neq 0$ and $\mathbf{m}_l \neq 0$ is carried out analogously and is not difficult.

2. "MASSIVE" PARTICLE

Let us consider the simplest case when the particle is entrained slightly by the turbulent motion. For this, the particle should be sufficiently massive (small b) so that the build-up period $1/a$ would greatly exceed the pulsation periods yielding the main contribution to **u** and **u'**. Large-scale pulsations in the stream possess the greatest period, and the condition $a < \omega_e$ corresponds to a "massive" particle, where ω_e is the frequency at the maximum of the turbulence spectrum. Since the force to entrain the small particle by a stream is proportional to its size d, and the inertial force to its volume, i.e., d^3, then as $d \to 0$ the approximation of a massive particle cannot be satisfied. This is reflected in a growing as $1/d^2$ and the condition $a < \omega_e$ being spoiled. At the same time, the upper bound on d is $d < l$. Combining both conditions we obtain

$$12b\sqrt{\mathrm{Re}} < d^2/l^2 < 1, \tag{5}$$

where $\mathrm{Re} = Lu_L/\nu$ is the Reynolds number for the stream, and u_L and L are the velocity scale and the size of the greatest pulsations.

The relations $\omega_e \sim u_L/L$, $L/l \sim \mathrm{Re}^{3/4}$, known from the theory of similarity of turbulent pulsations, were used in obtaining (5). By means of the condition (5) we have $\sqrt{\mathrm{Re}} < 1/12b$. For a particle in water we have $b \sim 1 \ldots 0.1$, i.e., $1/12b < 1$. Since the flow mode with $\mathrm{Re} < 1$ is not turbulent, then it is impossible to realize the case of a "massive" particle in a turbulent stream of water. Applied to a solid dust particle in air $b \sim 10^{-4}$, and $\mathrm{Re} < 10^6$ is necessary. For example, $0.3 < d/l < 1$, a very narrow range of sizes d, corresponds to "massive" particles for $\mathrm{Re} = 10^4$.

Upon compliance with condition (5), the ratio between the amplitude of particle vibration and the amplitude of liquid-particle oscillation (i.e., the elementary volume of the unperturbed medium) equals a/ω_e in order of magnitude and is small. Hence, the spatial dependence of the right sides of (1) and (4) can be considered smooth at distances on the order of the span of the vibrations r(t), which permits using an averaging method standard for equations of the type of (1). Assuming

$$\mathbf{r}(t) = \mathbf{R} + \mathbf{r}_\sim,$$

where $r_\sim(t)$ characterizes the oscillation and **R(t)** is the mean location of a particle with respect to the time $\sim 1/\omega_e$, let us expand the right side of (1) (we denote it henceforth by **f**) in a series in $\mathbf{r}_\sim$ near the average trajectory $\mathbf{r} = \mathbf{R}(t)$. In a first approximation we have

$$\bar{\mathbf{f}} = \langle \mathbf{f} \rangle + \langle (\mathbf{r}_\sim \nabla)\mathbf{f} \rangle;$$
$$\ddot{\mathbf{r}}_\sim + a\dot{\mathbf{r}}_\sim \simeq \mathbf{f}_\sim, \quad \mathbf{f}_\sim = \mathbf{f} - \langle \mathbf{f} \rangle, \tag{6}$$

where the angular brackets denote the average with respect to the time $\sim 1/\omega_e$ and with respect to realizations of turbulent pulsations for fixed $\mathbf{r} = \mathbf{R}$, i.e., the expressions in the angular brackets are the Euler characteristics of turbulent pulsations.

Let us take into account that the amplitude of pulsations of the magnitude $b\mathbf{u}'_\sim = b(\mathbf{u}' - \langle \mathbf{u}' \rangle)$ is much less than the amplitude of the pulsations $a\mathbf{u}_\sim = a(\mathbf{u} - \langle \mathbf{u} \rangle)$. Indeed,

$$\frac{b}{a}\frac{u'_\sim}{u_\sim} \sim \frac{d^2}{12\nu}\left(\omega_e + \frac{u_*}{L_*}\right) \sim \frac{1}{12\sqrt{\mathrm{Re}}}\frac{d^2}{l^2} + \frac{\mathrm{Re}_d}{12}\frac{d}{L_*} \ll 1, \tag{7}$$

where u_* and L_* are the characteristic amplitudes of the turbulent pulsations and the size of the inhomogeneous domain; $\mathrm{Re}_d = \sim u_* d/\nu$.

The quantity u_* is on the order of the dynamic velocity ("the friction velocity") in a turbulent boundary layer, and L_* is the characteristic layer thickness. It can hence be considered that $(u_* L_*/\nu \gtrsim 10)$. It can hence be considered that $f_\sim \approx a\mathbf{u}_\sim$. Using the spectral representation, we have from (6)

$$\mathbf{r}_\sim \approx \int_{-\infty}^{\infty} \frac{ae^{j\omega t}}{-\omega^2 + ja\omega} \mathbf{Z}(\mathbf{R}, d\omega),$$

where $\mathbf{z}_\sim$ is the random spectral amplitude of $\mathbf{u}_\sim$ in the interval $d\omega$. By virtue of the condition of stochasticity

$$\langle \mathbf{Z} \rangle = 0, \quad \langle \mathbf{Z}_i(\mathbf{r}, d\omega) Z_k^*(\mathbf{r}, d\omega') \rangle = \delta(\omega - \omega') W_{ik}(\mathbf{r}, \omega)\, d\omega d\omega',$$

where $\delta(\omega - \omega')$ is the delta function and W_{ik} is the spectral tensor:

$$\langle u_{\sim i}(\mathbf{r}, t)\; u_{\sim k}(\mathbf{r}, t) \rangle = \int_{-\infty}^{\infty} W_{ik}(\mathbf{r}, \omega)\, d\omega.$$

Let us assume for the estimates that the dependence of the spectrum density on ω has the very same form

$$W \sim k^4/(1+k^2)^3, \; k = \omega/\omega_e \tag{8}$$

for all the quantities needed. Such a dependence (see Secs. 3 and 5 in [7]) is interpolated by the spectral density $u^2_\sim$ for isotropic turbulence in the range from the largest vortices (the k^4 law) including the inertial subregion of the spectrum [for large k formula (8) does not differ too radically from the $k^{-5/3}$ law]. Taking account of (8), we obtain

$$\langle(\mathbf{r}_\sim \nabla)\mathbf{f}\rangle \approx \langle(\mathbf{r}_\sim \nabla) a\mathbf{u}_\sim\rangle \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{a^2 e^{j(\omega-\omega')t}}{-\omega^2 + ja\omega} \langle(\mathbf{Z}(\mathbf{R}, d\omega)\nabla) \times$$

$$\times \mathbf{Z}^*(\mathbf{R}, d\omega')\rangle = -\varkappa^2 \langle(\mathbf{u}_\sim \nabla)u_\sim\rangle = -\varkappa^2 \langle \mathbf{u}'_\sim\rangle, \; \varkappa^2 = \frac{k_a^2}{3}\frac{1+3k_a}{(1+k_a)^3}, \; k_a = a/\omega_e. \tag{9}$$

Let us take the average of the force $\mathbf{f}_\Omega$. We have

$$\mathbf{f}_\Omega = (b/2)\{[\langle\Omega\rangle \times \langle\mathbf{v}\rangle] + \langle[(\varphi_\sim - \psi_\sim) \times (\dot{\mathbf{r}}_\sim - \mathbf{u}_\sim)]\rangle\}. \tag{10}$$

The quantities $\varphi_\sim$ and $\dot{\mathbf{r}}_\sim$, which must be substituted into (10), will be calculated for fixed $\mathbf{r} = \mathbf{R}$. Taking account of the small oscillations therein is an effect of the next order of smallness in k_a. Also taking into account the inequality (7) and $\beta\psi'_\sim/\alpha\psi_\sim \ll 1$, which is similar, we obtain

$$\mathbf{v}_\sim = \dot{\mathbf{r}}_\sim - \mathbf{u}_\sim = \int_{-\infty}^{\infty} \frac{-j\omega e^{j\omega t}}{j\omega + a}\mathbf{Z}(\mathbf{R}, d\omega),$$

$$\Omega_\sim = \varphi_\sim - \psi_\sim = \frac{1}{2}\,\mathrm{rot}\left[\int_{-\infty}^{\infty} \frac{-j\omega e^{j\omega t}}{j\omega + \alpha}\mathbf{Z}(\mathbf{R}, d\omega)\right].$$

Hence, under the same assumptions as in the derivation of (9) and taking account of the vector identity $[\mathrm{rot}\,\mathbf{u}\times\mathbf{u}] = (\mathbf{u}\nabla)\mathbf{u} - \nabla(\mathbf{u}^2/2)$, we obtain

$$\langle[\Omega_\sim \times \mathbf{v}_\sim]\rangle = (\lambda/2)\left(\langle\mathbf{u}'_\sim\rangle - \nabla\frac{\langle u^2_\sim\rangle}{2}\right), \tag{11}$$

where

$$\lambda = 1 - \frac{1}{k_a + k_\alpha}\left[\frac{k_a^3 + 3k_a^4}{3(1+k_a)^3} + \frac{k_\alpha^3 + 3k_\alpha^4}{3(1+k_\alpha)^3}\right], \; k_\alpha = \alpha/\omega_e.$$

By virtue of the assumptions made, the quantity $[\langle\Omega\rangle \times \langle\mathbf{v}\rangle]$ turns out to be much less than the expression (11) and we will not write it down.

We consequently arrive at a description of the average motion of a massive particle:

$$\ddot{\mathbf{R}} + a\dot{\mathbf{R}} \approx a\langle\mathbf{u}\rangle + b\langle\mathbf{u}'\rangle - \varkappa^2\langle\mathbf{u}'_\sim\rangle - \frac{\lambda b}{4}\left(\nabla\langle u^2_\sim\rangle/2 - \langle\mathbf{u}'_\sim\rangle\right).$$

In many problems the turbulence characteristics vary relatively smoothly in space in the direction of the mean velocity $\langle\mathbf{u}\rangle$. In that case $\langle\mathbf{u}'_\sim\rangle \approx \langle\mathbf{u}'\rangle$. For a medium described by the Navier–Stokes equation with $\mathrm{div}\,\mathbf{u} = 0$ we hence have

$$\mathrm{rot}\,\langle\mathbf{u}'_\sim\rangle \approx \mathrm{rot}\,\langle\mathbf{u}'\rangle = \nu\Delta\,\mathrm{rot}\,\langle\mathbf{u}\rangle \neq 0.$$

This means that the pulsating force in (12) cannot be represented as a gradient of some effective potential in the general case.

The coefficients b and $\varkappa$ for forces of pulsating origin in (12) are small compared to the units of the magnitude of such conditions for a massive particle. Although $\varkappa/b \approx a/\sqrt{3}b\omega_e \sim 7\mathrm{Re}\,l^2/d^2 \gg 1$, but since $\varkappa$ is small, then $\varkappa^2 \ll b$ and $\varkappa^2 \gg b$ are possible. The condition $\varkappa^2 \ll b$ is equivalent to $b \ll (1/50\mathrm{Re})\,d^4/l^4$, which corresponds to an ultramassive particle. In the second (more actual) case when $b \gg (1/50\mathrm{Re})\,d^4/l^4$ [but the inequality (5) is satisfied], we have for the force of pulsating origin in (12)

$$\mathbf{f}_p \approx -\varkappa^2\langle\mathbf{u}'_\sim\rangle - \frac{\lambda b}{4}\nabla\frac{\langle u^2_\sim\rangle}{2}.$$

Despite the smallness of $b/\varkappa^2$, the second member remains here, since the gradient of the pulsation intensity in an inhomogeneous turbulent stream can greatly exceed the mean acceleration.

Therefore, knowledge of the spatial dependences of the Euler means $\langle\mathbf{u}\rangle$, $\langle\mathbf{u}^2_\sim\rangle$, and $\langle\mathbf{u}'\rangle$ permits investigation the drift of massive particles on the basis of (12).

3. LIGHT PARTICLE

A particle is entrained more strongly by turbulent motion as a grows, the right sides of (1) and (4) become sharply nonlinear, and the preceding results should undergo qualitative changes for $a/\omega_e > 1$.

Corresponding to the case of a strong increase is $a > \Omega_l$, where Ω_l characterizes the frequency per inner scale l in the Lagrange turbulence spectrum [i.e., in the spectrum of the field $u(r(t), t)$, where $\dot{r}(t) \equiv u$ is the velocity of a fluid particle]. The frequency $u_l/l \sim \omega_e\sqrt{Re}$ corresponds to the scale l in the Euler turbulence spectrum [i.e., in the spectrum of $\mathbf{u}(\mathbf{r}, t)$ for r = const]. Since $\Omega_l < u_l/l$, then the particle can knowingly be considered entrained ("light") if $a > \omega_e\sqrt{Re}$, which is equivalent to $d/l < \sqrt{b}$. All sufficiently small particles evidently fall under this condition. Such particles behave at each point almost as unperturbed fluid particles and the amplitude of the oscillations in the relative distance between the fluid and impurity particles is much less than the amplitude of the fluid particle vibrations. Let us use this circumstance to construct an approximate solution.

Let us examine the translational motion of a particle in some small time segment $(0, \Delta t)$. Let us assume

$$\mathbf{r}(t) = \mathbf{s} + \boldsymbol{\xi},$$

where $\mathbf{s} = \mathbf{s}(\mathbf{r}_0, t)$ is the path of the particle unperturbed by the fluid which is at the point $\mathbf{r}_0 = \mathbf{r}(0)$ at the initial time. For light particles the relative displacement $\xi(t)$ is mainly oscillatory and small for small Δt. Using the averaging procedure, we have in a first approximation

$$\overline{\mathbf{f}} = \langle \mathbf{f} \rangle_{La} + \langle (\boldsymbol{\xi}\nabla)\mathbf{f} \rangle_{La}, \tag{13}$$

where the brackets $\langle\ \rangle_{La}$ denote the means along the path $\mathbf{r} = \mathbf{s}(\mathbf{r}_0, t)$; the trajectories $\mathbf{s}(\mathbf{r}_0, t)$ are random in a turbulent stream and the average is carried out with respect to their statistical ensemble, i.e., the brackets $\langle\ \rangle_{La}$ are the mean Lagrangian characteristics. The equation for ξ is

$$\ddot{\boldsymbol{\xi}} + a\dot{\boldsymbol{\xi}} = b\mathbf{u}'_{\sim}(\mathbf{s}(t), t) - \frac{d\mathbf{u}_{\sim}(\mathbf{s}(t), t)}{dt}, \tag{14}$$

where $\mathbf{u}_{\sim} = \mathbf{u} - \langle \mathbf{u} \rangle_{La}$. Let us integrate (14). Since $\mathbf{u}'_{\sim} \approx d\mathbf{u}_{\sim}/dt$ in the times $\sim 1/a$, which are quite small for "light" particles, then we obtain

$$\boldsymbol{\xi} = (b = 1) \int_{-\infty}^{\infty} \frac{j\omega e^{j\omega t}}{-\omega^2 + ja\omega} \mathbf{Z}(\mathbf{r}_0, d\omega).$$

An exponential drop in time is characteristic for Lagrangian correlations, hence, we take the a dependence on ω of the form

$$W \sim 1/(\omega^2 + \omega_{La}^2).$$

for the spectral densities, where $\omega_{La} \sim 1/T_{La}$; T_{La} is the time of the drop in the Lagrangian correlations. Then

$$\langle (\boldsymbol{\xi}\nabla)\mathbf{f} \rangle_{La} \approx \langle (\boldsymbol{\xi}\nabla) a\mathbf{u}_{\sim} \rangle_{La} = (b-1)\varkappa_{La}^2 \langle (\mathbf{u}_{\sim}\nabla)\mathbf{u}_{\sim} \rangle_{La}, \quad \varkappa_{La}^2 = \frac{a}{a + \omega_{La}}.$$

The correlators $\langle \mathbf{u} \rangle$ are functions of the initial position of the particle $\mathbf{r}_0$. Dividing the whole t interval into small segments Δt, let us obtain the mean force (13) with its value of $\mathbf{r}_0$ in each.

In order to advance the analysis further, we must limit ourselves to the condition of smoothness of the turbulent field of pulsations, which is that if an integral Lagrangian scale T_{La} is selected as the time Δt, the displacement of fluid particles in the time $T_{La} = \int_t^{t+T_{La}} ds$ should be small, on the average, as compared with the distances within which the turbulence characteristics vary substantially. Since the fluid particles forget their prehistory during the time T_{La}, then the means in (13) become functions dependent on the stream properties at the site of the mean particle location in each interval T_{La}, i.e., functions of $\mathbf{R}(t)$. Hence, $\langle \mathbf{u} \rangle_{La}$ has the meaning of a mean displacement velocity of the center of gravity of particles located in a space with center at the point $\mathbf{r} = \mathbf{R}$, $\langle \mathbf{u}' \rangle$ has the meaning of their mean acceleration, etc.* These charac-

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*Let us note that $\langle \mathbf{u}(\mathbf{r}, t) \rangle_{La} = \langle \mathbf{u}(\mathbf{r}, t) \rangle$ and $\langle \mathbf{u}'(\mathbf{r}, t) \rangle_{La} \equiv \langle \mathbf{u}'(\mathbf{r}, t) \rangle$ for fluid particles at the point $\mathbf{r}$ at the time t by the definition of the Lagrangian means. We operate with several other characteristics: the velocity and acceleration of fluid particles averaged with respect to the initial positions in a finite volume and with respect to a finite time – the characteristic pulsation period.

teristics agree with the Euler means within the limits of homogeneous isotropic turbulence.

Let us take the average of the force $\mathbf{f}_\Omega$ by an analogous method. We have

$$\mathbf{v} = (b-1)\int_{-\infty}^{\infty} \frac{j\omega e^{j\omega t}}{j\omega + a}\mathbf{Z}(\mathbf{r}_0, d\omega),$$

$$\mathbf{\Omega} = \frac{\beta - 1}{2}\,\mathrm{rot}\left[\int_{-\infty}^{\infty} \frac{j\omega e^{j\omega t}}{j\omega + \alpha}\mathbf{Z}(\mathbf{r}_0, d\omega)\right].$$

Hence,

$$\mathbf{f}_\Omega = -\frac{b(1-b)(1-\beta)}{4}\lambda_{La}\left(\nabla\frac{\langle \mathbf{u}_\sim^2\rangle_{La}}{2} - \langle \mathbf{u}'_\sim\rangle_{La}\right),$$

where

$$\lambda_{La} = \frac{1 + \dfrac{2}{\omega_{La}}\dfrac{\alpha a}{\alpha + a}}{1 + \dfrac{\alpha + a}{\omega_{La}} + \dfrac{\alpha a}{\omega_{La}^2}} \approx \frac{2\omega_{La}}{\alpha + a}.$$

Consequently, we arrive at the following description for the average motion of a light particle:

$$\dot{\mathbf{R}} = \langle \mathbf{u}\rangle_{La} + \frac{b}{a}\langle \mathbf{u}'\rangle_{La} - \frac{1-b}{a}\varkappa_{La}^2\langle \mathbf{u}'_\sim\rangle_{La} - \frac{b(1-b)(1-\beta)}{4a}\lambda_{La}\left(\nabla\frac{\langle u_\sim^2\rangle_{La}}{2} - \langle \mathbf{u}'_\sim\rangle_{La}\right). \tag{15}$$

The term $\sim\ddot{\mathbf{R}}$ is omitted, since $a/\omega_{La} \gg 1$. In the limit $a \to \infty$ (i.e., $d \to 0$) we obtain $\dot{\mathbf{R}} = \langle \mathbf{u}\rangle_{La}$, i.e., the mean drifts of impurity particles and fluid particles agree, as they should. In contrast to (12), no smallness condition is imposed on the density factor b (and β) in (15). Let us note that the coefficient for the force of rotational origin is relatively small, as before, since $b\lambda_{La}/\varkappa_{La}^2 \sim b(\omega_{La}/a) \lesssim \omega_{La}/a \ll 1$.

Thus, for light particles the mean force from the stream is expressed in terms of the pulsation characteristics by a formula similar in structure to (12) for the massive particles, but now they are Lagrange means. Considerably less is ordinarily known about these characteristics than about the Euler means, and this makes a quantitative analysis of the question difficult. One of the essential differences in the motion of light and massive particles is that the mean velocity $\langle \mathbf{u}\rangle_{La}$ in an inhomogeneous turbulent stream has a significant component in the direction perpendicular to the average flow velocity $\langle \mathbf{u}_\sim\rangle$, which is related to the very manner of the existence of an inhomogeneous turbulent stream.

4. PARTICLE IN AN INHOMOGENEOUS TURBULENT LAYER

Let us apply the theory to the case of a particle in a turbulent incompressible medium whose field characteristics **u** are stationary and vary in just one direction (the x_2 axis) perpendicular to the mean flow velocity (the x_1 axis). Such conditions can be some idealizations of real flows of a liquid and gas in tubes, near flat walls, in a turbulent medium, and in other cases.

Let us consider the behavior of a massive particle. Its average motion $R = (x_1, x_2, x_3)$ is determined by (12). For the geometry assumed $\langle \mathbf{u}\rangle = (U, 00)$, $\langle \mathbf{u}'\rangle = \langle \mathbf{u}'_\sim\rangle = (d/dx_2)\,\langle u_2\mathbf{u}_\sim\rangle$, where $\langle u_2u_3\rangle = 0$, $U = U(x_2)$. The equation of longitudinal particle motion is

$$\ddot{x}_1 + a(\dot{x}_1 - U) = -(\varkappa^2 - b - \lambda b/4)d\langle u_1u_2\rangle/dx_2. \tag{16}$$

where $u_1 = (\mathbf{u}_\sim)_1$. The quantity $\langle u_1u_2\rangle$ is the turbulent shear stress. Because of $\langle u_1u_2\rangle$ transverse transport of turbulent velocities in the stream is realized and grad $\langle u_1u_2\rangle$ is directed towards diminishing velocity pulsations so that $d\langle u_1u_2\rangle/dx_2$ equals zero in the zone of maximum turbulence and has different signs on both sides of this zone.

To be specific, let us examine the flow in a long pipe. The $\langle u_1u_2\rangle$ profile is presented in the Fig. 1 (according to the data of Laufer [7]), where D is the pipe diameter, U_0 is the mean velocity on the axis, u_* is the friction velocity. The x_2 axis is directed along the radius from the wall, and all the curves fall to zero as $x_2 \to 0$. At the wall $\langle u_1u_2\rangle = 0$, then $\langle u_1u_2\rangle$ takes on negative values, reaches a minimum, and then grows in the core of the stream by dropping to zero in magnitude. In the stream core $(d/dx_2)\,\langle u_1u_2\rangle \approx 2u_*^2/D$, and this quantity has its greatest (negative) value in the zone $x_2u_*/\nu \sim 10$–15 and is on the order of $10^3\cdot u_*^2/D \sim U_0^2/D$. For an air stream ($\nu = 0.15$ cm/sec) moving at a velocity 15 m/sec along a pipe of 10 cm diameter, Re = 10^5, $u_* = 75$ cm/sec, $2u_*^2/D \simeq 10^3$ cm/sec^2 $\sim$ g, $U_0^2/D \simeq 2\cdot10^5$ cm/sec^2 $\sim 2\cdot10^2$ g, where g is the acceleration of gravity.

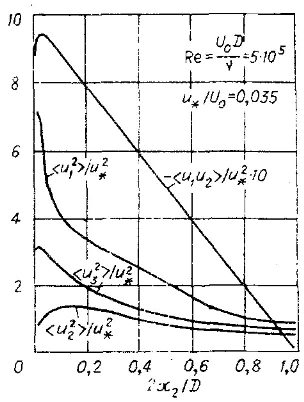

Fig. 1

Let us note that the shape of the turbulence spectrum W for flow in a pipe varies with the change in distance to the axis and differs for different values of W. The assumption (8) is spoiled, and, strictly speaking, (12) is not legitimate for this case. However, the turbulence spectrum enters the problem in integrated form; hence, the equations are slightly responsive to deviations from the law (8). It is important that the Reynolds number for the particle was small at all points of the section and its relaxation frequency a was much less than the frequency of concentration of the pulsation spectrum **u**. But this is a constraint on the frequency and not on the stream.

Let us consider what (16) yields for the case of a particle in a pipe. For $\varkappa^2 > b + \lambda b/4$ the force on the right in (16) is positive in the layer near the wall and accelerates the particle,† but in the stream core the force is directed opposite to the mean flow velocity and decelerates the particle. The particles exert the reverse effect on the stream: at the wall they decelerate the flow but accelerate it in the stream core. This occurs because of the energy acquired by the particles from the turbulent pulsations.

Let us clarify the physical reason for this interesting behavior. According to the Navier–Stokes equation

$$\left\langle \rho \frac{d}{dt} u_1 \right\rangle = \rho \frac{d}{dx_2}\langle u_1 u_2 \rangle = -\frac{dp}{dx_2} + \rho\nu \frac{d^2}{dx_2^2} U.$$

The mean force in an isolated fluid volume from the side of its circumference is on the right and is due to the pressure p and the viscosity. The quantity $-\rho d\langle u_1u_2\rangle/dx_2$ can be interpreted as the inertial force, which is of pulsation origin and is due to vibrations of the isolated volume (on the average it is not accelerated $dU/dt \equiv 0$). This force is none other than the resultant Reynolds stress in the volume. The same circumference acts on the impurity particle (with a correction for the effect of the apparent mass), but the inertial force differs from $-\rho d\langle u_1, u_2\rangle/dx_2$. Cancellation of both kinds of forces does not occur; consequently, a force additional to the Stokes force appears in (16). Part of it $b(d/dx_2)\langle u_1 u_2\rangle = b\langle u_1'\rangle$ is due to the pressure gradient in the surrounding fluid, the term $\sim\varkappa^2$ is related to the translational motion inertia, and the term $\sim\lambda$ is related to the rotational motion inertia. For $\varkappa^2 < b + \lambda b/4$ the particle is ultramassive, its pulsations are quite small, and the force from the side of the circumference predominates and we have the customary direction of action of the forces. For $\varkappa^2 > b + \lambda b/4$, the pulsation forces predominate; they agree with the direction of action $-\rho d\langle u_1u_2\rangle/dx_2$ and evoke the above-mentioned effect.

Let us examine the transverse particle drift. It is independent of the longitudinal [the converse is false as is seen from (16)] and is described by the equation

$$\ddot{x}_2 + a\dot{x}_2 = -\left(\varkappa^2 - b - \frac{\lambda b}{4}\right)\frac{d}{dx_2}\langle u_2^2\rangle - \frac{\lambda b}{4}\frac{d}{dx_2}q^2 = f_2, \tag{17}$$

where $q^2 = {}^1/_2\langle u_\sim^2\rangle = {}^1/_2\langle u_1^2 + u_2^2 + u_3^2\rangle$. As is seen from Fig. 1, the functions $\langle u_2^2\rangle$ and q^2 near the wall have different characters of variation. In the viscous sublayer $q^2 \sim 0.1\, u_*^2 y^2$, $y = x_2u_*/\nu$ the function $\langle u_2^2\rangle$ drops more rapidly than y^2 as $x_2 \to 0$, and, therefore more rapidly than q^2 (see Fig. 7.35 in [7]), outside the viscous sublayer the contribution from $\sim dq^2/dx_2$ in (17) diminishes, and $f_2 \simeq -\varkappa^2(d/dx_2)\langle u_2^2\rangle$ for $\varkappa^2 \gg b$. Therefore, f_2 changes sign for $\varkappa^2 \gg b$ in the zone of maximum $\langle u_2^2\rangle$, i.e., this zone is the "watershed" for

†Let us note that in a viscous sublayer ($x_2u_*/\nu \lesssim 1$) the pulsating forces can exceed the Stokes force aU, since $(\varkappa^2/aU)$ for $x_2u_*/\nu = 1$, $d\langle u_1u_2\rangle/dx_2 \sim 10^{-3}$ Rea/ω e.

the average transverse particle displacement. Corresponding to it is $y \sim 5 \cdot 10^2$; for large y the particles drift toward the pipe axis and for small y, towards the wall.

The function $\langle u_2^2 \rangle$ is approximated in the range $1 < y < 5 \cdot 10^2$ by the formula $\langle u_2^2 \rangle = u_*^2 [y/(y + y_0)]^2$, where $y_0 \approx 10$ [5]. For $\varkappa^2 \gg b$ in this zone (17) becomes

$$\ddot{y} + a\dot{y} + \frac{1}{\tau_0^2} \frac{y}{(1 + y/y_0)^3} \simeq 0, \tag{18}$$

where $\tau_0 = (y_0/\sqrt{2\varkappa})(\nu/u_*^2)$. The quantity $a\tau_0 \approx y_0(\nu\omega_e/u_*^2) \sim (y_0/\mathrm{Re})(U^2/u_*^2)$ is independent of the particle parameters and is small $[(y_0/\mathrm{Re})(U^2/u_*^2) \lessapprox 0.1]$ in a broad band $\mathrm{Re} > 10^4$. For $a\tau_0 \ll 1$ the drift is determined mainly by the time τ_0. For small $a\tau_0$ we obtain an estimate of the time $\tau(y)$ for a particle, which is at the point y at the time t = 0 and is at rest, to reach the viscous layer from (18):

$$\tau y = (\pi/2)\tau_0(1 + y/y_0)^{3/2} + (a\tau_0/3)\tau_0(y/y_0)^3 + \ldots.$$

The second member is commensurate with the first for $y \simeq y_0\sqrt{3\pi/2a\tau_0} \sim 10^2$. For $y < y_0$, $\tau(y)$ depends slightly on y and the order τ_0. For a particle of density $\rho_0 = 10$ g/cm³ in an air stream of the example presented above we have $b = 2 \cdot 10^{-4}$, $\omega_e \simeq 3 \cdot 10^2$ sec⁻¹, $l \sim \nu/u_* \simeq 2 \cdot 10^{-3}$ cm. Let us set the particle diameter equal to $2 \cdot 10^{-2}$ cm; then $a = 90$ sec⁻¹. We obtain $\tau_0 \approx 10^{-1}$ sec. A massive particle falling freely from a height y = 50 flies in the stream for a time $\tau_g = \sqrt{(2y/g)(\nu/u_*)} \approx 1.4 \cdot 10^{-2}$ sec, i.e., considerably longer.

Let us discuss the case of a light particle. Without having available the information needed about the Lagrange means which are in (15), let us limit ourselves just to several remarks. For the flow symmetry under consideration the velocity $\langle \mathbf{u} \rangle_{La}$ has both longitudinal and transverse components $\langle u \rangle_{La} = (U_{La}, V_{La}, 0)$. It is possible to take $U_{La} \approx U(x_2)$. The transverse velocity V_{La} is due to the inconstancy of dU/dx_2. In the zone of turbulence generation (here $|dU/dx_2|$ is large) $V_{La}(x_2)$ is such that the fluid particles drift to layers relative to the flow at rest. But $V_{La}(x_2)$ has an opposite direction in these rest zones; hence, the fluid particles depart to the generation zone on the average by quenching their vortical energy. There they gather energy and are again on the path; because of such convection a stationary turbulence mode is maintained. Let us note that the relative drift of the fluid particles, i.e., the difference between $\langle \mathbf{u} \rangle_{La}$ and $\langle \mathbf{u} \rangle$, is caused by forces of pulsating origin – the Reynolds stress gradients. The transverse acceleration of the fluid particle is

$$\langle u_2'(\mathbf{r}, t) \rangle_{La} = \langle u_2(\mathbf{r}, t) \rangle = \frac{d}{dx_2} \langle u_2^2 \rangle.$$

Corresponding to the zone of maximum generation of turbulence in application to flow in a pipe is $y = y_G \sim 15$. For $y < y_G$ we should have $V_{La}(x_2) > 0$. Evidently, $V_{La}(x_2) = 0$ in the viscous sublayer. In the region $y \approx 10$, apparently $V_{La} \sim 0.1 U_*$ (V_{La} turns out to be of such an order in the extremum zone $d\langle u_1 u_2 \rangle/dx_2$ of a free turbulent jet (see Secs. 5 and 6 in [7])). For $y > y_G$, $V_{La}(x_2) < 0$. Let us put $\langle \mathbf{u}' \rangle_{La} = \langle \mathbf{u}' \rangle$ in (15) (this is legitimate only for slightly inhomogeneous turbulence; hence, we limit ourselves to a qualitative picture of the motion). Then $\langle \mathbf{u}'_L \rangle_{La} \approx \langle \mathbf{u}' \rangle - (d/dx_2)(V_{La} \langle \mathbf{u} \rangle_{La})$ and for small b we have for longitudinal particle motion

$$\dot{x}_1 - U \approx -\frac{\varkappa_{La}^2}{a} \frac{d}{dx_2} (\langle u_1 u_2 \rangle - UV_{La}).$$

It follows from the discussion presented above that the profile of the change in $-UV_{La}$ is approximately similar to the profile of $\langle u_1 u_2 \rangle$. The action of both members in the parentheses is hence added. The direction of the action agrees with that which occurred for the massive particles. Hence, the reverse reaction of the particle on the medium is analogous in character. Since $\varkappa_{La}$ for light particles, then the specific force per unit mass of the particle is now considerably greater.

For small b we have for transverse motion

$$\dot{x}_2 - V_{La} \approx -\frac{\varkappa_{La}^2}{a} \frac{d}{dx_2} \langle u_2^2 \rangle - \frac{\lambda_{La} b}{4a} \frac{d}{dx_2} q^2. \tag{19}$$

Terms whose smallness is on the order of V_{La}^2/q^2 are omitted in the equations. The essential difference between this equation and (17) is the presence of the member V_{La}. For $y \approx 10$, $V_{La} \sim 0.1 u_*$ and the terms on the right in (19) are much less than this quantity in estimates so that the drift of soft particles together with the fluid particles to the stream core occurs here. The drift velocity $V_{La}(x_2)$ drops with the approach to the viscous sublayer. It can be shown that the drop in $V_{La}(x_2)$ should be more rapid than the rate of the drop in the Reynolds stress. Hence, the role of the terms on the right in (19) becomes predominant near the wall – here $x_2 < 0$. For $y \sim 1$ and $b \ll 1$

$$-\dot{x}_2 \approx -\frac{\varkappa_{\mathrm{La}}^2}{a} \frac{d}{dx_2} \langle u_2^2 \rangle \approx \frac{2\varkappa_{\mathrm{La}}^2}{y_0^2} \frac{u_*^3}{a\nu} \sim 2 \cdot 10^{-2} \frac{u_*^3}{\nu g} \frac{g}{a},$$

where g/a is the rate of gravitational precipitation of particles. For the example with the air stream presented $2 \cdot 10^{-2} u_*^3/\nu g \approx 60$, i.e., $|\dot{x}_2| \gg g/a$.

Let us note that the estimates made are insufficient for a comparison with test, and test, in particular, verifies the fact of accelerated settling of particles on the wall from the turbulent stream (see [3-5, 12-15], for example†). The fact is that individual particles are not observed in experiments (this is complex) but the rate of deposition of a set of particles on the wall or their distribution over the stream section is. Moreover, either requires the solution of the diffusion problem, i.e., the solution of the diffusion equation or the kinetic equation taking account of pulsating forces and some boundary conditions. This important problem for applications is beyond the scope of this paper.

Therefore, it is shown that disordered motions of a particle, in which it is entrained by the turbulent medium, result in significant effects, cumulative over many periods of turbulent pulsations. The relationship between the Stokes frequency of particle relaxation in the stream and the characteristic frequencies of the turbulence spectrum hence plays an important part. The extreme cases of "light" particles (among these are all particles of small enough size) and of "massive" particles have been investigated.

Equations of the form (12) and (15), simple in structure, in which the Euler and Lagrange turbulence characteristics, respectively, enter, have been obtained for the average particle motions in both cases. The equations merge well, which permits assumption of a possible interpolation of the results in the range of parameters where $a \sim \omega_{\mathrm{e}} - \omega_{\mathrm{La}}$. A direct analysis in this intermediate region is complicated because of the lack of a small parameter in the problem. Characteristic for this region is the growth of the contribution to the average motion from the disordered particle rotations. The rotation factor must be taken into account for light and massive particles only in the zone where the turbulence is sharply inhomogeneous.

The nature of the particle drift in a turbulent stream between parallel walls has been examined on the basis of (12) and (15). The theory has been constructed for weakly inhomogeneous turbulence; hence, its application to this problem does not assure high accuracy of the estimates, as has been noted. However, the qualitative behavior is visibly reflected correctly. Among the general qualitative deductions presented here are the following. 1. The significance of the forces of pulsating origin and their commensurability (converted to unit particle mass) with the magnitudes of the characteristic accelerations in the stream. 2. The presence of a watershed for the particle drift motions transverse to the stream. There exists a zone where dense (small b) particles "are attracted" to the wall, but a zone where the particles "are repelled" from the wall lies sufficiently far from the viscous layer ($y \gtrsim 5 \cdot 10^2$). 3. For particle drift longitudinal to the stream (for which b is small) the particles leading the flow in the zone near the viscous sublayer and lagging the flow in the stream core are characteristic. Since the effect is caused by forces coming from the medium, then there exists a reverse reaction on the stream. Particles turning out to be in the flow core accelerate it and those being near the wall, decelerate it. The resultant effect depends on the particle distribution over the stream section. The presence of a small ("passive") impurity in an inhomogeneous turbulent stream does not thereby reduce, as is sometimes considered, to just a change in the viscosity of the medium. In addition, the inverse transformation of turbulent medium motions into directed motions occurs.

Additional forces of pulsating origin on the particle in a turbulent medium appear for the same physical reason for which effective (sometimes called fictitious) forces expressed in terms of the Reynolds stress act on fluid particles (i.e., small volumes of the medium). Intrinsically, the Reynolds stresses cause a difference between the average fluid particle motion and the Euler motion of the medium. This circumstance has been taken into account in analyzing the Lagrangian characteristics in Sec. 4. Development of this question permits a new approach to the description of inhomogeneous turbulence. However, this is the subject of a separate discussion.

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†Although the mechanism considered for the phenomenon is apparently sufficiently effective, it is not unique. Many authors tend to the fact that the main mechanical factor (there are others, the particle electrification factor, for example) is that particles are sometimes ejected from the stream by turbulent gusts and fly through the viscous layer to the wall by inertia.

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